$$\hat{\theta}_{ML} = \arg\max_{\theta} \left(\prod_{k=1}^{N} p(\vec{x}_k ; \theta) \right)$$

$$L(\vec{\theta}) = \ln \left(\prod_{k=1}^{N} p(\vec{x}_k ; \theta) \right)$$

$$\frac{\partial L(\theta)}{\partial \theta} = \sum_{k=1}^{N} \frac{1}{p(\vec{x}_k ; \theta)} \frac{\partial}{\partial \theta} \left(p(\vec{x}_k ; \theta) \right) = 0$$

P1 Let $p(x_k, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(x_k - \mu)^2}{2\sigma^2} \right)$. Compute $\hat{\sigma}^2_{ML}$

$$p(x_k, \sigma^2) = (2\pi)^{-\frac{1}{2}} (\sigma^2)^{-1/2} \exp\left(- \frac{(x_k - \mu)^2}{2\sigma^2} \right)$$

$$L(\sigma^2) = \ln \left(\prod_{k=1}^{N} (2\pi)^{-1/2} (\sigma^2)^{-1/2} \exp\left(- \frac{(x_k - \mu)^2}{2\sigma^2} \right) \right)$$

$$= \sum_{k=1}^{N} \ln \left((2\pi\sigma^2)^{-1/2} \right) - \sum_{k=1}^{N} \frac{(x_k - \mu)^2}{2\sigma^2}$$

$$= -\frac{N}{2} \ln (2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{k=1}^{N} (x_k - \mu)^2$$

$$\hat{\sigma}^2_{ML} : \frac{dL(\sigma^2)}{d\sigma^2} = 0$$

$$\frac{dL(\sigma^2)}{d\sigma^2} = -\left(\frac{N}{2}\right) \left(\frac{1}{\cancel{2\pi}\,\sigma^2} \right) (\cancel{2\pi}) + \frac{1}{2} \left(\frac{1}{\sigma^4} \right) \sum_{k=1}^{N} (x_k - \mu)^2$$

$$\frac{dL(\sigma^2)}{d\sigma^2} = 0$$

$$\left(\frac{N}{\cancel{2}} \right) \left(\frac{1}{\sigma^2} \right) = \left(\frac{1}{\cancel{2}} \right) \left(\frac{1}{\sigma^4} \right) \left(\sum_{k=1}^{N} (x_k - \mu)^2 \right)$$

$$\Rightarrow \boxed{\hat{\sigma}^2_{ML} = \frac{1}{N} \sum_{k=1}^{N} (x_k - \mu)^2}$$

Properties

(1) ML estimate is asymptotically <u>unbiased</u>

$$\lim_{N \to \infty} E\left[\hat{\theta}_{ML}\right] = \theta_0$$

(2) ML estimate is asymptotically <u>consistent</u>

$$\lim_{N \to \infty} \text{prob}\left(\|\hat{\theta}_{ML} - \theta_0\| \leq \epsilon\right) = 1$$

Alternatively, a strong condition:

$$\lim_{N \to \infty} E\left[\|\hat{\theta}_{ML} - \theta_0\|^2\right] = 0$$

(3) ML estimate is asymptotically <u>efficient</u> that it achieves Cramer-Rao lower bounds.

(4) The pdf of ML estimate as $N \to \infty$ approaches the <u>Gaussian distribution</u> with mean θ_0.